

# Kinetic Theory Approach to the SK Spin Glass Model with Glauber Dynamics

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## Abstract

I present a new method to analyze Glauber dynamics of the Sherrington-Kirkpatrick (SK) spin glass model. The method is based on ideas used in the classical kinetic theory of fluids. I apply it to study spin correlations in the high temperature phase ( $T \geq T_c$ ) of the SK model at zero external field. The zeroth order theory is equivalent to a disorder dependent local equilibrium approximation. Its predictions agree well with computer simulation results. The first order theory involves coupled evolution equations for the spin correlations and the dynamic (excess) parts of the local field distributions. It accounts qualitatively for the error made in the zeroth approximation.

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The dynamical properties of the Sherrington–Kirkpatrick (SK) spin glass model [1] have been a subject of continuous interest in recent years [2]. However, almost all the theoretical studies considered Langevin dynamics of the soft-spin version of the SK model [3–5]. The soft-spin version, while showing very interesting dynamical properties [5], lacks the original motivation of the SK model: neither its statics nor its dynamics is exactly solvable [6]. The Glauber dynamics of the SK model was studied by Sommers [7]. He recovered the results found previously for the Langevin dynamics. Sommers’ method was criticized by Łusakowski [8] and its validity is uncertain [9]. Recently a novel approach to Glauber dynamics of spin glasses [10,11] has been proposed by Coolen, Sherrington, and coworkers (CS). The simple version of their theory [10] describes very well the order parameter *flow direction* above the de Almeida-Thouless (AT) [12] line but misses the *slowing down* which sets in when the former line is approached from above. The more advanced version [11] agrees well with the simulation data for short times but it remains to be seen whether it predicts divergent relaxation times at and below the AT line.

Here I reconsider the Glauber dynamics of the SK model. The original motivation for this work was to improve the simple CS theory [10]. However, the resulting method is very different from that of CS.

CS tried to derive a general description of the SK spin glass dynamics. The theory presented here is more restricted: I study time-dependent spin correlations *in equilibrium* in the high temperature phase ( $T \geq T_c$ ) at zero external field. The main motivation is simplicity: it is possible to derive explicit results for these correlations, and it is easy to perform accurate computer simulations that allow testing the theoretical predictions.

Following an approach used in kinetic theory [13], I express the correlation functions in terms of a distribution that satisfies the master equation and a specific initial condition. Next I propose a series of approximations for this distribution that are motivated by the approximations used in the kinetic theory [14]. Successive approximations gradually include *dynamic* many-spin correlations. The static correlations are retained at every step.

The approximations are formulated for a given sample of the coupling constants. The

averaging over the samples is postponed until after the resulting evolution equations are solved.

The simplest (0th order) approximation is equivalent to a *disorder dependent* version of the local equilibrium approximation [15]. It leads to very simple equations of motion for the spin correlations: the relaxation matrix is a product of a relaxation rate (kinetic coefficient),  $\tau^{-1}$ , that is finite at the transition temperature,  $T_c$ , and an inverse matrix of equilibrium spin correlations,  $A_{ij}$ , or the Hessian of the Thouless-Anderson-Palmer (TAP) [16] free energy. The Hessian acquires zero eigenvalues at  $T_c$  [17]. This results in a mean-field-like critical slowing down of the time-dependent correlations when  $T_c$  is approached from above and an algebraic decay  $\sim t^{-1/2}$  at  $T_c$ . A comparison with the simulation data shows that the zeroth order approximation is surprisingly accurate.

The first order approximation takes into account *dynamic* correlations between spins and the distributions of the local fields acting on these spins: it includes time-delayed Onsager reaction fields.

The first order approximation accounts qualitatively for the error made in the zeroth order: the predicted *difference* between the full correlations and the zeroth order approximation is about 40% of the simulation result.

I now sketch the derivation of the results. I consider the Glauber dynamics for the SK model of a spin glass. The time evolution is given by the master equation for the spin probability distribution  $P(\boldsymbol{\sigma}; t)$ ,

$$\partial P(\boldsymbol{\sigma}; t)/\partial t = - \sum_i (1 - S_i) w_i(\boldsymbol{\sigma}) P(\boldsymbol{\sigma}; t). \quad (1)$$

Here  $\boldsymbol{\sigma} \equiv \{\sigma_1, \dots, \sigma_N\}$  denotes the spin configuration,  $S_i$  is the spin-flip operator,  $S_i \sigma_i = -\sigma_i$ , and  $w_i(\boldsymbol{\sigma})$  is the transition rate,  $w_i(\boldsymbol{\sigma}) = (1 - \sigma_i \tanh(\beta h_i))/2$ , with  $h_i$  being a local magnetic field acting on the  $i$ th spin,  $h_i = \sum_{j \neq i} J_{ij} \sigma_j$ . The  $J_{ij}$  are the exchange coupling constants that are quenched random variables distributed according to the symmetric distribution  $P(J_{ij}) \sim \exp(-J_{ij}^2/(2J^2/N))$ .

I study the time-dependent correlations of the total magnetization in equilibrium,

$(1/N) [\langle m(t)m(0) \rangle_{eq}]$ . Here  $m(t) = \sum_i \sigma_i(t)$  is the fluctuation of the magnetization (for  $T \geq T_c$  at zero external field  $\langle \sigma_i \rangle_{eq} \equiv 0$ ), the angular brackets  $\langle \dots \rangle_{eq}$  denote the equilibrium ensemble average, and the square brackets  $[\dots]$  denote the sample averaging over the distribution of  $J_{ij}$ 's.

I perform the sample averaging at the very last stage of the analysis. Therefore for the most part I deal with sample dependent quantities like  $\langle \sigma_i(t)m(0) \rangle_{eq}$ . This is analogous to the TAP [16] analysis of the equilibrium SK model and to early work [1,18] on the Glauber dynamics of the SK model. It is different from the CS approach and also from most of the other approaches to both Langevin [3–5] and Glauber [7] dynamics.

The correlations  $\langle \sigma_i(t)m(0) \rangle_{eq}$  are defined in terms of a conditional distribution  $P(\boldsymbol{\sigma}; t | \boldsymbol{\sigma}')$  and the equilibrium distribution  $P_{eq}(\boldsymbol{\sigma})$  [19]. I define a distribution  $P_m(\boldsymbol{\sigma}; t)$ ,

$$P_m(\boldsymbol{\sigma}; t) \equiv \sum_{\boldsymbol{\sigma}'} P(\boldsymbol{\sigma}; t | \boldsymbol{\sigma}') \left( \sum_j \sigma'_j \right) P_{eq}(\boldsymbol{\sigma}'). \quad (2)$$

The distribution  $P_m$  satisfies the master equation (1) and the initial condition [20]

$$P_m(\boldsymbol{\sigma}; t=0) = P_{eq}(\boldsymbol{\sigma}) \sum_i \sigma_i.$$

The time-dependent spin correlations in equilibrium  $\langle \sigma_i(t)m(0) \rangle_{eq}$  can be calculated as averages over  $P_m$ ,

$$\langle \sigma_i(t)m(0) \rangle_{eq} = \sum_{\boldsymbol{\sigma}} \sigma_i P_m(\boldsymbol{\sigma}; t) = \langle \sigma_i \rangle(t). \quad (3)$$

Hereafter  $\langle \dots \rangle(t)$  denotes average over the time-dependent distribution  $P_m$ . In the following I propose a series of approximations for this distribution.

In the zeroth approximation I assume that  $P_m(t)$  can be expressed in terms of the single-spin averages,  $\langle \sigma_i \rangle(t)$ . More precisely, I assume that  $P_m$  has the same form as an equilibrium distribution for the system in an external field with the field chosen in such a way that the single spin averages have correct values. Explicitly,

$$P_m(\boldsymbol{\sigma}; t) \approx P_{eq}(\boldsymbol{\sigma}) \sum_i \sigma_i b_i(t), \quad (4)$$

where fields  $b_i(t)$ ,  $i = 1, \dots$  satisfy the following equations,

$$\langle \sigma_i \rangle (t) = \sum_k \langle \sigma_i \sigma_k \rangle_{eq} b_k(t). \quad (5)$$

Solving Eq. (5) for the  $b_k(t)$  I get

$$P_m(\boldsymbol{\sigma}; t) \approx P_{eq}(\boldsymbol{\sigma}) \sum_{ij} \sigma_i A_{ij} \langle \sigma_j \rangle (t). \quad (6)$$

Here the matrix  $A_{ij}$  is the inverse of the matrix of the equilibrium spin correlations,  $\sum_j A_{ij} \langle \delta \sigma_j \delta \sigma_k \rangle_{eq} = \delta_{ik}$ . Note that  $A_{ij}$  is identical to the Hessian of the TAP free energy [16].

The ansatz (6) is similar to the local equilibrium approximation introduced by Kawasaki [15]. The new element of this work is to use the local equilibrium approximation for the *disorder dependent* distribution  $P_m$ .

To derive the equations of motion for the spin averages I start from the exact evolution equations,

$$\partial \langle \sigma_i \rangle (t) / \partial t = - \langle \sigma_i \rangle (t) + \langle \tanh(\beta h_i) \rangle (t). \quad (7)$$

Then I use ansatz (6) to calculate the averages at the right-hand-side (RHS) of Eqs. (7) and obtain

$$\frac{\partial \langle \sigma_i \rangle (t)}{\partial t} = \left( -1 + \langle \tanh(\beta h_i) \sigma_i \rangle_{eq} \right) \sum_j A_{ij} \langle \sigma_j \rangle (t). \quad (8)$$

According to Eqs. (8) the dynamics of the spin correlations follows a van Hove mean-field-like picture: the relaxation matrix is a product of the relaxation rate,  $\tau^{-1} = 1 - \langle \tanh(\beta h_i) \sigma_i \rangle_{eq}$ , and the inverse matrix of the spin correlations (Hessian),  $A_{ij}$ . Each of Eqs. (8) contains an Onsager correction term [1] that has been introduced phenomenologically in early works [1,18]. Within the zeroth order theory the correction term is *instantaneous*: the reaction field at a given time depends on the value of the spin average at the same time.

The relaxation rate can be calculated with the help of the equilibrium probability distribution of the local fields,  $P_{eq}(h)$  [21]. Numerical evaluation shows that at  $T_c$  the relaxation

rate is finite. On the other hand the Hessian,  $A_{ij}$ , acquires zero eigenvalues at the transition temperature [17] and this fact leads to a mean-field-like critical slowing down as  $T_c$  is approached from above. Moreover, at  $T_c$  I obtain asymptotically  $[\langle\sigma_i\rangle(t)] \sim t^{-1/2}$ .

In the high temperature phase at zero external field the Hessian is known explicitly:

$$A_{ij} = -\beta J_{ij} + \delta_{ij}(1 + (\beta J)^2). \quad (9)$$

It follows that the evolution equations (8) are almost identical to those derived in the original SK paper [1]. The solution has the same form as the solution of the SK equations if the time scale of SK is rescaled by factor  $\tau$ .

In Fig. 1 I compare predictions of the zeroth order theory with numerical simulations of the SK model at the transition temperature. 10 samples of  $N = 10000$  spins each have been simulated using algorithm of Mackenzie and Young [22]. Very long equilibration time of 10000 Monte Carlo steps per spin (MCS) was used. Subsequently the data for the time-dependent correlation function  $(1/N) \sum_i \langle\sigma_i(t)\sigma_i(0)\rangle_{eq}$  were collected [23] and averaged over different time origins [24]. The figure indicates that the zeroth order theory is quite accurate: its predictions differ from the simulation data by less than 11%. In Fig. 1 I also plot predictions of the second order Sommers theory. They were obtained by solving explicitly Eq. (18) of Ref. [7], using the fluctuation-dissipation theorem to get the Laplace transform of the correlation function, and finally inverting the Laplace transform numerically [25].

In Fig. 2 I plot the difference between the simulation data and the predictions of the zeroth order theory. It is clear that the zeroth order approximation is not exact. This fact can also be seen from an analysis of the short time behavior of the spin correlations: the zeroth order theory reproduces exactly the first time derivative at  $t=0$  but *not* the second and higher order derivatives.

To improve upon the zeroth order theory it is necessary to go beyond the local equilibrium approximation and include *dynamic* correlations [14,26]. It follows from the physics of the SK model and from the analysis of the short time expansion of the time-dependent spin

correlations that the first additional set of variables to be included are the dynamic (excess) parts of the local field distributions,  $\delta P_i(h; t)$  [27]. They are defined as the differences between the true distributions and their values in the local equilibrium ensemble (6),

$$\begin{aligned} \delta P_i(h; t) &= \langle \delta(h - h_i) \rangle(t) \\ &\quad - \sum_{jk} \langle \delta(h - h_i) \sigma_j \rangle_{eq} A_{jk} \langle \sigma_k \rangle(t). \end{aligned} \quad (10)$$

At  $t=0$  the excess parts vanish,  $\delta P_i(h; t=0) = 0$ .

To derive equations of motion for the spin averages and the excess parts of the local field distributions I need an approximate expression for the distribution  $P_m$  in terms of  $\langle \sigma_i \rangle(t)$  and  $\delta P_i(h; t)$ . I assume that  $P_m$  has the following form:

$$P_m(\boldsymbol{\sigma}; t) \approx P_{eq}(\boldsymbol{\sigma}) \left( \sum_{ij} \sigma_i A_{ij} \langle \sigma_j \rangle(t) + \sum_{ij} \int dh \delta^e(h - h_i) \int dq C_{ij}(h, q) \delta P_j(q; t) \right). \quad (11)$$

Here  $\delta^e(h - h_i)$  is the microscopic expression for the excess part of the local field distribution,

$$\delta^e(h - h_i) = \delta(h - h_i) - \sum_{jk} \langle \delta(h - h_i) \sigma_j \rangle_{eq} A_{jk} \sigma_k, \quad (12)$$

and  $C_{ij}(h, q)$  is the inverse “matrix” of the correlations of the excess local field distributions,  $\sum_j \int dq C_{ij}(h, q) \langle \delta^e(q - h_j) \delta^e(p - h_k) \rangle_{eq} = \delta_{ik} \delta(h - p)$ .

The form of the distribution (11) is motivated by approximations used in the kinetic theory [14,26]. Briefly, to get (11) I assume that  $P_m(t)$  has the same form as an equilibrium distribution for the system in the presence of external perturbations that are chosen in such a way that, at a given time, the single spin averages and the excess parts of the local field distributions are  $\langle \sigma_i \rangle(t)$  and  $\delta P_i(h; t)$ , respectively. Now I will show that with the help of (11) one can describe qualitatively the difference between the predictions of the local equilibrium approximation and the simulation data.

First I derive equations of motion for the spin averages. I start from the exact equations (7), use (11) to calculate averages, and obtain the following equations of motion,

$$\frac{\partial \langle \sigma_i \rangle(t)}{\partial t} = -\frac{1}{\tau} \sum_j A_{ij} \langle \sigma_j \rangle(t) + \int dh \tanh(\beta h) \delta P_i(h; t). \quad (13)$$

Next I derive equations of motion for the excess parts of the local field distributions. To this end I start from exact evolution equations,

$$\frac{\partial \delta P_i(h; t)}{\partial t} = \left\langle \left[ \sum_i w_i(\boldsymbol{\sigma}) (1 - S_i) \delta^e(h - h_i) \right] \right\rangle (t), \quad (14)$$

use the distribution (11), and get

$$\begin{aligned} \frac{\partial \delta P_i(h; t)}{\partial t} = & - \left[ (\beta J)^2 \tanh(\beta h) - \beta h \right] P_{eq}(h) \frac{\partial}{\partial t} \langle \sigma_i \rangle (t) \\ & + \frac{(\beta J)^2}{\tau} \frac{\partial}{\partial \beta h} \int dq [P_{eq}(h) \delta(h - q) - P_{eq}(h) P_{eq}(q)] \frac{\partial}{\partial \beta q} \int dq' C_{ii}(q, q') \delta P_i(q'; t), \end{aligned} \quad (15)$$

where  $P_{eq}(h)$  is the equilibrium local field distribution. To derive (15) I keep two-point equilibrium correlations, *e.g.*,  $\langle \sigma_i \sigma_j \rangle_{eq}$ , but I neglect *higher order connected* correlations involving *different* lattice sites, *e.g.*,  $\langle \delta^e(h - h_i) \delta^e(q - h_j) \rangle_{eq}$  for  $i \neq j$  [28].

According to Eqs. (13) the reaction field consists of two parts: an instantaneous reaction, proportional to the spin average at the same time, and a time-delayed reaction. It follows from (15) that the delayed reaction field acting on  $i$ th spin at a given time depends on the values of this spin at earlier times.

Solving formally Eqs. (15) and substituting the result into Eqs. (13) one gets a set of effective equations of motion that involve only the single-spin averages. These equations have the same *structure* as the memory function equation derived recently by Kawasaki for dissipative stochastic systems [29]: the inverse frequency,  $\tau$ , gets renormalized by inclusion of the dynamic correlations.

To derive explicit results for the time-dependent correlations I solve the integro-differential equation (15) approximating  $\delta P_i(h; t)$  by a finite sum of basic functions.

In Fig. 2 I compare theoretical predictions for the difference between the full spin correlations and the local equilibrium result against the simulation data. The agreement is quantitative for short times (the first order theory reproduces exactly first two time derivatives of the spin correlations at  $t=0$ ) and qualitative at long times.



It is evident from Fig. 2, and it can be shown theoretically, that the first order theory is not exact. Glauber dynamics of the SK model is more complicated than statics: in addition to (possibly time-delayed) Onsager reaction fields other *dynamic* correlations have to be included.

In summary, I have shown that the ideas of the classical kinetic theory of fluids can be used to analyze Glauber dynamics of the SK spin glass model. The very simple disorder dependent local equilibrium approximation leads to quantitatively accurate predictions for the spin correlations, at least in the high temperature phase of the SK model. This fact suggests that *a* disorder dependent approximation might be a good starting point in the search for a general theory of the SK model dynamics. Secondly, the error made in the zeroth approximation can be accounted for by including time-delayed Onsager reaction fields. The resulting theory can be improved further by incorporating more complicated dynamic correlations. Finally, the method presented here can be generalized to dynamics of neural networks and other Ising-like systems.

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## FIGURES

FIG. 1. Spin correlation function at the transition temperature  $T_c$ . Crosses: Glauber dynamics simulation data; dashed line: the zeroth order theory (local equilibrium approximation); solid line: the first order approximation; dotted line: the second order Sommers' theory.

FIG. 2. The difference between the full spin correlation function and the local equilibrium approximation at the transition temperature  $T_c$ . Crosses: Glauber dynamics simulation data; solid line: the first order approximation.